Scattering of dislocated wave fronts by vertical vorticity and the Aharonov-Bohm effect. II. Dispersive waves

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Previous results on the scattering of surface waves by vertical vorticity on shallow water are generalized to the case of dispersive water waves. Dispersion effects are treated perturbatively around the shallow water limit, to first order in the ratio of depth to wavelength. The dislocation of the incident wave front, analogous to the Aharonov-Bohm effect, is still observed. At short wavelengths the scattering is qualitatively similar to the nondispersive case. At moderate wavelengths, however, there are two markedly different scattering regimes according to whether the depth is smaller or larger than $\sqrt{3}$ times capillary length. In the latter case, dispersion and advection may compensate leading to a spiral interference pattern. The dislocation is characterized by a parameter that depends both on phase and group velocity. The validity range of the calculation is the same as in the shallow water case: wavelengths small compared to vortex radius, and low Mach number. The implications of these limitations are carefully considered. [S1063-651X(99)18710-5]

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I. INTRODUCTION

In the preceding paper [1], hereafter referred to as I, we studied the scattering of surface waves by a stationary vertical vortex in the long wavelength approximation: surface tension was neglected and the fluid depth was supposed to be small compared to wavelength. This is also called the shallow water approximation. There were two motivations for the study of shallow water waves scattering. First, they are nondispersive waves, like acoustic waves in fluids, and it was plausible that a generalization of calculations for sound scattering by vorticity [2] was feasible. Second, it was a first attempt towards a quantitative confirmation of the heuristic approach of Berry *et al.* [3]. The aim of this paper is to go beyond this approximation.

In actual experimental situations [4] the shallow water limit is hard to obtain and, if a quantitative comparison with experiment is desired, it becomes necessary to take into account the finite depth and the surface tension. The main difference between surface waves in shallow water and in deeper water lies in the fact that in the latter case dispersion effects are important: there are two length scales, one associated with depth and the other with surface tension, which are responsible for wave velocity depending on wavelength. In this paper we seek to describe the scattering of surface waves by vorticity in terms of a single differential equation in which surface elevation is the only dependent variable. This is possible in a perturbative treatment away from the shallow water case, and we present here results that correspond to first-order corrections.

As in I, we consider the scattering of surface waves by a stationary vortex, in the limit of a small Mach number (the velocities of fluid particles are small by comparison with the phase velocity of the waves), $M \ll 1$, and a large wave number k, i.e., $\beta \equiv ka \gg 1$ where a is a typical length associated with the vortex flow. The product $M\beta$ is assumed to be of the order of 1. In Sec. II, we derive from the full hydrodynamic set of equations an approximation valid to order $O(M)$ [or $O(\beta^{-1})$]. First, equations are linearized for small surface perturbations around a steady vertical vortex and then higher-order terms in M and β^{-1} are discarded. We shall pay particular attention to the orders of magnitude of the different terms, and will justify the neglect of dissipative effects. The recovery of the shallow water results is subtle since it involves taking the singular limit of vanishing surface tension. There appears a partial differential equation [Eq. (2.25) below] that contains a squared Laplacian, and it is reduced to our previous result, Eq. (2.7) of I, in the shallow water limit, i.e., when the layer's depth is small and surface tension is negligible.

The solution of Eq. (2.25) is given in Sec. III. The results given by Eqs. (3.4) and (3.20) seem much more complicated than the similar shallow water results, Eqs. (4.5), (4.9), and (4.10) of I. However, this complexity is essentially algebraic, and actually the physical results are rather similar, except when dispersive effects are closely balanced by advection to yield a spiral pattern for the scattered waves. The wave front dislocation is characterized by a parameter α that is a generalization of the one in I, and tends towards it smoothly in the shallow water limit. In the dispersive case, α depends on both the phase and group velocity of the waves. We give a perturbative justification of the heuristic argument of Berry *et al.* [3]. The behavior of the scattered wave, however, depends strongly on the ratio of depth to capillary length. We also exhibit two different behaviors, depending on the relative values of the fluid depth and capillary length. At each important step in the calculations, we verify that the shallow water limit is recovered. However, the partial differential equations, Eqs. (2.25) and (2.7) of I differ in the order of differentiation, with surface tension appearing as a coefficient of the highest derivative term in Eq. (2.25); the limit of null surface tension is thus singular. Graphical illustrations of the solution are given in Sec. IV for various values of the dislocation parameter α and for fluid depth larger and smaller than capillary length.

II. WATER WAVES IN INTERACTION WITH A VERTICAL VORTEX

Equations for an incompressible fluid of equilibrium depth h, free surface $h+\eta(x,y,t)$ with origin of vertical coordinates $(z=0)$ at the bottom, lying in a (uniform) gravitational field g are

$$\partial_t \boldsymbol{V}+\boldsymbol{V}\cdot\boldsymbol{\nabla}\boldsymbol{V}=-\frac{1}{\rho}\boldsymbol{\nabla}P-g\hat{z}, \tag{2.1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{V}=0, \tag{2.2}$$

where $\boldsymbol{V}$ is the fluid velocity, P is the pressure, and ρ is the (constant) density.

We neglect viscous dissipation. This is justified if the viscous attenuation time $T_{\rm diss}$ of the wave is greater than a period $T_{\rm wave}$. The attenuation times for shallow water waves is [5]

$$T_{\rm diss}=\frac{\sinh 2kh}{k\left(\frac{\mu\nu}{2\rho}\right)^{1/2}}, \tag{2.3}$$

where μ is the dynamic viscosity of the fluid, h is the depth, k is the wave number, and ν is the wave frequency. In the case of water, $\mu=0.01$ g/cm s, $g=981$ cm/s^2, and $\rho=1$ g/cm^3. Below we justify that our approximation of the dispersion relation is valid up to $kh\sim 0.8$. Taking experimentally reasonable values such as $h=1$ mm and a wavelength of about 1 cm, we get $T_{\rm diss}/T_{\rm wave}\sim 9$ for 1 cm, and $T_{\rm diss}/T_{\rm wave}\sim 7$ for 0.5 cm. It is thus reasonable to neglect viscosity. As a matter of fact, we do not expect qualitative changes due to viscosity, apart from a decrease in the wave amplitude, which is, of course, not predicted in our calculations.

Boundary conditions are that fluid elements at the free surface of the fluid remain there, that pressure has a discontinuity that is exactly compensated by surface tension, and that there is no vertical velocity at the bottom:

$$z=h+\eta:\quad V_z=\partial_t\eta+\boldsymbol{V}_\perp\cdot\boldsymbol{\nabla}_\perp\eta, \tag{2.4}$$

$$z=h+\eta:\quad P=-\tau\nabla_\perp^2\eta, \tag{2.5}$$

$$z=0:\quad V_z=0, \tag{2.6}$$

where τ is the surface tension, $\boldsymbol{V}_\perp$ is the horizontal velocity, and $\nabla_\perp$ is the horizontal gradient. We are interested in small perturbations $(\boldsymbol{v},p_1,\eta_1)$ around a steady, axially symmetric, vertical vortex $(\boldsymbol{U},P_0,\eta_0)$. The vertical vortex is given by the (divergenceless) flow $\boldsymbol{U}=U_0(r)\hat{\theta}$ in cylindrical coordinates (r,θ,z), where $(\hat{r},\hat{\theta},\hat{z})$ are the unit vectors in the radial, tangential, and vertical direction, respectively.

The zero-order situation, $\boldsymbol{v}=0$, gives

$$P_0=-\rho g z+p_0(x,y,t),\quad \frac{U_0^2}{r}=\frac{1}{\rho}\partial_r p_0. \tag{2.7}$$

Given a specific function U_0 this is integrated at once. Concerning boundary conditions, the third boundary condition (2.6) is satisfied identically. The first boundary condition (2.4) says that the surface deformation is independent of polar angle θ, and the second boundary condition (2.5) gives the free surface η_0 in terms of the pressure:

$$p_0=\rho g\eta_0-\tau\boldsymbol{\nabla}_\perp^2\eta_0. \tag{2.8}$$

Writing $\boldsymbol{v}=(\boldsymbol{u},w)$ and neglecting terms quadratic in v we have the equations to the order of 1:

$$(\partial_t+\boldsymbol{U}\cdot\boldsymbol{\nabla}_\perp)\boldsymbol{u}+\boldsymbol{u}\cdot\boldsymbol{\nabla}_\perp\boldsymbol{U}=-\frac{1}{\rho}\nabla_\perp p_1, \tag{2.9}$$

$$(\partial_t+\boldsymbol{U}\cdot\boldsymbol{\nabla}_\perp)w=-\frac{1}{\rho}\partial_z p_1 \tag{2.10}$$

$$\boldsymbol{\nabla}_\perp\cdot\boldsymbol{u}+\partial_z w=0. \tag{2.11}$$

Similarly, the boundary conditions to the order of 1 are [6]

$$z=h+\eta:\quad w=(\partial_t+\boldsymbol{U}\cdot\nabla_\perp)\eta_1+\boldsymbol{u}\cdot\nabla_\perp\eta_0, \tag{2.12}$$

$$z=h+\eta:\quad p_1=\rho g\eta_1-\tau\nabla_\perp^2\eta_1, \tag{2.13}$$

$$z=0:\quad \partial_z p_1=0, \tag{2.14}$$

where we used Eq. (2.10) to obtain the third boundary condition. Taking the divergence of Eqs. (2.9) and (2.10), and using Eq. (2.11) gives

$$\boldsymbol{\nabla}_\perp^2 p_1+\partial_{zz}p_1=-2\rho(\boldsymbol{\nabla}_a U_b)(\boldsymbol{\nabla}_b u_a). \tag{2.15}$$

Up to now, these equations are exact for *linear* surface waves interacting with a static vortex. It is the fact that linear waves exist that provides us with another parameter, the phase velocity, with which to compare U. We will now simplify the problem by using the following two approximations: First, the typical velocity of the vortical flow U_0 is supposed to be much less than the *phase* velocity of the wave c_ϕ. Second, the wavelength λ is supposed to be much smaller than a typical length associated with the vortex a. In practice, a will be the core radius of the vortex, and we assume $ka\gg 1$ where $k\equiv 2\pi/\lambda$ is the wave vector. We will denote formally the small quantities by ϵ. We thus assume $U_0/c_\phi\equiv M=O(\epsilon)$, where M will be called the Mach number in analogy with acoustics, and $ka=O(1/\epsilon)$, and we search for corrections of order ϵ to the wave equation without permanent vortical flow. To get the relative importance of the terms that appear in the differential equations, we will use the following estimates:

$$\boldsymbol{\nabla}_\perp f_0\sim\frac{f_0}{a},\quad \partial_t f_1\sim\nu f_1,\quad \nabla_\perp f_1\sim k f_1,\quad \partial_z f_1\sim k f_1, \tag{2.16}$$

where f_0 is any scalar quantity referring to the vortical flow, f_1 is any scalar quantity referring to the surface waves, and ν is the wave frequency. We have assumed that the length scales for vertical and horizontal variations of surface waves are the same, as in the absence of the vortex. This may be derived by injecting the other scalings in Eqs. (2.9), (2.11), and (2.10).

With those estimates, we get from Eq. (2.10) that

$$\frac{k}{\rho} p_1 \sim \nu w \Rightarrow p_1 \sim \rho c_\phi w. \tag{2.17}$$

Injecting this result in Eq. (2.15), the order of magnitude of the left-hand side is $k^2 p_1 = k^2 \rho c_\phi w$, whereas the right-hand side is of the order of $\rho(U_0/a)kw = k^2 \rho c_\phi w (U_0/c_\phi)(1/ka)$; it is thus negligible, being of order $O(\epsilon^2)$, and Eq. (2.15) is replaced by

$$\nabla_\perp^2 p_1 + \partial_{zz} p_1 = 0, \tag{2.18}$$

which is the same Laplace equation as in the problem of water waves without the vortex; it has the big advantage of being autonomous and linear in the pressure so that separation of variables can be attempted.

An estimate of the surface elevation η_0 for the vortex flow may be obtained from Eqs. (2.8) and (2.7); it reads

$$\eta_0 \sim \frac{U_0^2}{g}\left(1 + \frac{l_c^2}{a^2}\right)^{-1}, \tag{2.19}$$

where we have introduced the *capillary length* $l_c \equiv \sqrt{\tau/\rho g}$. For water, $\tau = 74$ dyn/cm, so that $l_c \approx 0.32$ cm and the effect of surface tension on the surface deformation of a vortex of size $a \approx 1$ cm is quite small, of the order of 1%. The surface wave elevation from Eqs. (2.13) and (2.17) reads

$$\eta_1 \sim \frac{c_\phi w}{g}(1 + k^2 l_c^2)^{-1}, \tag{2.20}$$

with $k^2 l_c^2$ of the order of 1. In the following, we take $\eta_1 \sim c_\phi w/g$, which is numerically inexact but adequate for the order of magnitude considerations. In Eq. (2.12), the respective orders of magnitude of the different terms are $\boldsymbol{U} \cdot \boldsymbol{\nabla}_\perp \eta_1 \sim M(\partial_t \eta_1)$ and $\boldsymbol{u} \cdot \boldsymbol{\nabla}_\perp \eta_0 \sim (M^2/\beta)(\partial_t \eta_1)$, so that the relevant approximation for Eq. (2.12), valid to $O(\epsilon)$, reads

$$z = h: \quad w = (\partial_t + \boldsymbol{U} \cdot \nabla_\perp)\eta_1. \tag{2.21}$$

In this equation, we neglect η in comparison with h. If we write $p_1(h+\eta_0) = p_1(h) + \delta p_1$ and use Eq. (2.19), we get $\delta p_1/p_1 \sim k\eta_0 \sim M^2$, so that δp_1 is indeed negligible and the boundary condition is to be taken at $z = h$. The same is true for Eq. (2.13).

Let us use now these approximate equations to describe the propagation of surface waves in the vortical flow. We will consider almost shallow water waves, that is, the next order in the small parameter kh of the calculations of I. In this limit, the pressure is found as a power series in the vertical coordinate z, which, inserting boundary condition (2.14), reads

$$p_1(r,\theta,z,t) = \sum_{m=0}^{\infty} (-1)^m z^{2m} \frac{\nabla_\perp^{2m}\Pi}{(2m)!}. \tag{2.22}$$

We introduce the notation $D_t \equiv \partial_t + \boldsymbol{U} \cdot \boldsymbol{\nabla}_\perp$. Taking only the leading-order terms in the small parameter kh in Eq. (2.22), applying the differential operator D_t to Eq. (2.21), and taking Eq. (2.10) for $z = h$, we get

$$D_t^2 \eta_1 = \frac{h}{\rho} \boldsymbol{\nabla}_\perp^2 \Pi - \frac{h^3}{6\rho} \boldsymbol{\nabla}_\perp^4 \Pi. \tag{2.23}$$

Applying $\boldsymbol{\nabla}_\perp^2$ to Eq. (2.13) for $z = h$ and using Eq. (2.22) at the same order, we get

$$gh \boldsymbol{\nabla}_\perp^2 \eta_1 - \frac{\tau h}{\rho} \boldsymbol{\nabla}_\perp^4 \eta_1 = \frac{h}{\rho} \boldsymbol{\nabla}_\perp^2 \Pi - \frac{h^3}{2\rho} \boldsymbol{\nabla}_\perp^4 \Pi. \tag{2.24}$$

The surface tension term is considered under the assumption that the capillary length is of the same order of magnitude as the depth of the fluid layer. In this case, those two equations are valid up to order $O[(kh)^2]$. It is thus legitimate to replace the pressure by its value at order $O(1)$, $\Pi = \rho g \eta_1$, in the term $\propto \nabla_\perp^4 \Pi$, which has the highest derivative. Eliminating the pressure in the resulting equations, we get the final result: a dispersive wave equation for surface elevation η_1 that is analogous to Eq. (2.7) of I in the shallow water case. It reads

$$gh \boldsymbol{\nabla}_\perp^2 \eta_1 + \left(\frac{1}{3} gh^3 - \frac{\tau h}{\rho}\right) \boldsymbol{\nabla}_\perp^4 \eta_1 - D_t^2 \eta_1 = 0. \tag{2.25}$$

This equation includes the leading-order correction to the shallow water case. It is valid under the same assumptions (see I) concerning wavelength and fluid velocity. It describes the scattering of surface waves over water whose depth is small but not negligible with respect to wavelength, when the wavelength is small compared to the vortex size, when the velocity of the vortex flow is much less than the phase velocity of the waves, and when the waves are of small amplitude.

Without the vortex, when $U = 0$ and $\partial_t = D_t$, plane progressive waves of the form

$$\eta_1 \propto e^{i(\nu t - k_\perp \cdot r_\perp)}$$

exist provided frequency ν and wave vector $k \equiv |k_\perp|$ are related through the dispersion relation

$$\nu^2 = ghk^2 + \left(\frac{\tau h}{\rho} - \frac{1}{3} gh^3\right) k^4, \tag{2.26}$$

which is the approximation to order $O[(kh)^3]$ of the well-known dispersion relation for capillary-gravity waves [5].

The wave dispersion is thus characterized by a dimensionless parameter δ defined by

$$k^2\left(\frac{\tau}{\rho g} - \frac{h^2}{3}\right) = \frac{1}{\delta}. \tag{2.27}$$

It is positive for $h < \sqrt{3} l_c$, and negative otherwise. We shall consider both cases. In order to be consistent with our approximations, namely, that the fourth-order term in Eq. (2.25) be a small correction to the other two, the absolute value of δ must be large, and the shallow water limit corresponds to $|\delta| \to \infty$. For positive δ, the phase and group velocity read

$$c_\phi^2 = gh \frac{1+\delta}{\delta}, \quad c_{\mathrm{g}} = \frac{gh}{c_\phi} \frac{2+\delta}{\delta} \quad (\delta > 0), \tag{2.28}$$

whereas for negative values of δ they read

$$c_\phi^2 = gh\frac{|\delta|-1}{|\delta|}, \quad c_g = \frac{gh}{c_\phi}\frac{|\delta|-2}{|\delta|} \quad (\delta<0). \quad (2.29)$$

The full dispersion relation for water waves [5] is either convex or concave at small depth, depending on the sign of δ. The crossover point $h=\sqrt{3}l_c$, derived from the approximate relation (2.26), separates two regions of opposite convexity. Both may be experimentally accessible. The approximation of the hyperbolic tangent is better than 1% for $kh<0.5$, and better than 5% for $kh<0.8$. It is thus easy to stay in the small depth limit, $\tanh(kh)\approx kh-(kh)^3/3$, while keeping the wavelength small in comparison with the vortex radius. Using a fluid with high surface tension like water leads to a positive δ, that is, $h<\sqrt{3}l_c$, whereas the same experiment with a fluid of small surface tension will give a negative δ.

III. SCATTERING OF DISLOCATED WAVES BY A VORTEX

We will now proceed exactly as in Sec. IV of I. Inside the vortex, the equation for the radial functions $\tilde{\eta}_{1n}$ factorizes exactly as

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}+(k_+)^2\right]\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}+(k_-)^2\right]\tilde{\eta}_{1n}=0, \quad (3.1)$$

with

$$(k_\pm)^2 \equiv \frac{1}{2}k^2\delta\left(-1\pm\sqrt{1+\frac{4(\nu-n\omega/2)^2}{ghk^2\delta}}\right) \quad (\delta>0) \quad (3.2)$$

or

$$(k_\pm)^2 \equiv \frac{1}{2}k^2|\delta|\left(1\pm\sqrt{1-\frac{4(\nu-n\omega/2)^2}{ghk^2|\delta|}}\right) \quad (\delta<0). \quad (3.3)$$

The two differential operators in Eq. (3.1) commute, and the four independent solutions of this fourth-order equation are thus given by the two pairs of solutions of the two corresponding second-order differential equations.

Taking the shallow water limit $\delta\to\infty$, we get for positive δ that k_+ tends toward the value of k_n corresponding to the shallow water case (see Eq. (4.4) of I) as it should, since this case must be recovered as a limiting case. The other constant k_- comes from the fact that Eq. (3.1) is a fourth-order differential equation, unlike Eq. (4.4) of I. Its limit for $\delta\to\infty$ is singular, reflecting the fact that surface tension τ multiplies the highest derivative term in differential equation (2.25). The respective role of k_+ and k_- are exchanged for negative δ.

From Eq. (3.2) we see that when δ is positive k_+ is real whereas k_- is imaginary for all n, whereas for negative δ the two wave vectors $k_\pm$ are real for small n and complex for large n. For positive δ, Eq. (3.1) has Bessel and Neumann functions as solutions, together with hyperbolic Bessel and hyperbolic Neumann functions. The Neumann and hyperbolic Neumann functions must be discarded because of regularity at the origin. For negative δ, we take Bessel and Neumann functions of a complex argument, and discard the Neumann functions to ensure regularity at the origin. Thus

$$\eta_1 = \mathrm{Re}\left[\sum_n \left(a_n\frac{J_n(k_n r)}{J_n(k_n a)}+b_n\frac{X_n(\kappa_n r)}{X_n(\kappa_n a)}\right)e^{i(n\theta-\nu t)}\right], \quad (3.4)$$

where the a_n and b_n are undetermined coefficients in both cases, and where we have introduced the notation

$$k_+\equiv k_n, \quad k_-\equiv i\kappa_n, \quad X_n\equiv I_n, \quad (\delta>0), \quad (3.5)$$

$$k_-\equiv k_n, \quad k_+\equiv \kappa_n, \quad X_n\equiv J_n, \quad (\delta<0). \quad (3.6)$$

Outside the vortex for $r>a$ dropping terms of order M^2, we get that Eq. (2.25) may be written in the factorized form

$$O_+O_-\tilde{\eta}_{1n}=0, \quad O_\pm \equiv \mathcal{L}-\frac{m_\pm^2}{r^2}+q_\pm^2, \quad (3.7)$$

where $\mathcal{L}\equiv d^2/dr^2+(1/r)(d/dr)$, provided the unknown coefficients m_+, m_-, q_+, and q_- satisfy the following relations:

$$(\mathcal{L}):\Rightarrow q_+^2+q_-^2=-\delta k^2, \quad (3.8)$$

$$(1):\Rightarrow q_+^2q_-^2=-(\delta+1)k^4, \quad (3.9)$$

$$\left(\frac{1}{r^2}\right):\Rightarrow m_+^2q_-^2+m_-^2q_+^2=-\delta k^2n^2-\frac{\delta k^2}{gh}\frac{2\Gamma\nu n}{2\pi}, \quad (3.10)$$

$$\left(\frac{\mathcal{L}}{r^2}\right):\Rightarrow m_+^2+m_-^2=2n^2, \quad (3.11)$$

$$\left(\frac{1}{r^3}\frac{d}{dr}\right):\Rightarrow m_-^2=n^2, \quad (3.12)$$

$$\left(\frac{1}{r^4}\right):\Rightarrow m_+^2m_-^2-4m_-^2=n^4-4n^2. \quad (3.13)$$

Here we have indicated on the left the portion of the linear differential operator that leads to each condition. There are six equations for only four unknowns and obviously they cannot be simultaneously satisfied in general. The last two equations, Eqs. (3.12) and (3.13), correspond to terms that are negligible at large distance from the vortex. If we compare them to $\mathcal{L}^2$, they are smaller than $1/\beta^2$ because $r>a$. Accordingly, we are justified in ignoring these two equations in our approximation $\beta\gg1$ and we solve Eqs. (3.8)–(3.11), that gives

$$q_+^2=k^2>0, \quad q_-^2\equiv(iq)^2=-k^2(1+\delta)<0,$$
$$(m_\pm)^2=n^2\pm2n\alpha \quad (\delta>0), \quad (3.14)$$

$$q_+^2\equiv q^2=(|\delta|-1)k^2>0, \quad q_-^2=k^2>0,$$
$$(m_\pm)^2=n^2\mp2n\alpha \quad (\delta<0), \quad (3.15)$$

$$\alpha \equiv \frac{\Gamma \nu}{2\pi} \frac{1}{gh+2(\tau/\rho - gh^2/3)hk^2} = \frac{\Gamma\nu}{2\pi}\frac{1}{c_\phi c_g}, \quad (3.16)$$

where we used Eq. (2.26) to write the last equality. It is important to note that the index $n^2+2n\alpha$ is always associated to the incident wave vector k. We will comment further on this result after Eq. (3.19). From now on, we define

$$m_+ \equiv \sqrt{n^2+2n\alpha}, \quad m_- \equiv \sqrt{n^2-2n\alpha}, \quad (3.17)$$

so that in the negative δ case m_- (respectively m_+) is associated with q_+ (respectively q_-), as shown by Eq. (3.15).

The dimensionless parameter α is defined in close analogy with I. We can write $\alpha = M\beta(c_\phi/c_g)$, which may be of the order of 1, which $M \ll 1$ and $\beta \gg 1$. This parameter has the same physical interpretation as in the shallow water case (see below): it gives the amount of dislocation for the wave fronts far from the vortex. This calculation provides an explicit confirmation, in a perturbation expansion near the shallow water case, of the intuitive result of Berry *et al.* [3].

The two differential operators $O_\pm$ in Eq. (3.7) do not commute. Using the usual notation $[\cdot,\cdot]$ for the commutator of two operators, we get

$$[O_+,O_-] = (m_+^2 - m_-^2)\left[\mathcal{L}, \frac{1}{r^2}\right] = 4(m_+^2-m_-^2)\left(\frac{1}{r^4} - \frac{1}{r^3}\frac{d}{dr}\right), \quad (3.18)$$

which is small, of the same order as the neglected terms, Eqs. (3.12) and (3.13), and will also be neglected. Thus, in the same approximation, for positive δ the solution of Eq. (3.7) is a linear combination of Bessel, Neumann, hyperbolic Bessel, and hyperbolic Neumann functions, because q_+ is real and q_- is imaginary. Since the hyperbolic Bessel function diverges at infinity, it must be discarded. For negative δ, the solution is a linear combination of Bessel and Neumann functions of argument kr and qr. Since the wave number q is that of a scattered wave, we discard the Bessel function of qr, keeping only the outgoing wave from the vortex.

Following Berry *et al.* [3] as in I, we write the surface elevation outside the vortex in the form

$$\eta_1 = \mathrm{Re}(\eta_{AB} + \eta_R), \quad (3.19)$$

where η_{AB} is defined exactly as in the previous case I. It does not depend on the sign of δ, which is physically obvious because the amount of dislocated wave front is linked to the circulation of the vortex, not to the curvature of the dispersion relation. Thus $m_+ = \sqrt{n^2+2n\alpha}$ is always the index of the functions involving the wave vector k. The other term of Eq. (3.19) depends on the sign of δ, which is also physically clear since they represent the wave scattered by the vortex. They read

$$\eta_R = \sum_n \left(d_n \frac{H^1_{m_+}(kr)}{H^1_{m_+}(ka)} + e_n \frac{Y_{m_-}(qr)}{Y_{m_-}(qa)}\right) e^{i(n\theta - \nu t)}. \quad (3.20)$$

Depending on the sign of δ, we have the following definitions:

$$Y_{m_-} \equiv K_{m_-}, \quad q = k\sqrt{1+\delta} \quad (\delta > 0), \quad (3.21)$$

$$Y_{m_-} \equiv H^1_{m_-}, \quad q = k\sqrt{|\delta|-1} \quad (\delta < 0). \quad (3.22)$$

The coefficients a_n, b_n, c_n, d_n, and e_n are defined so that they denote the amplitude of the wave components at the vortex boundary $r=a$. In order to obtain these coefficients, and since Eq. (2.25) is of the order of 4, the continuity of $\tilde{\eta}_1$ and its first three derivatives at $r=a$ is required, which gives four relations. The fifth and last boundary condition comes from the asymptotic behavior of η at infinity. We require that the asymptotics of η_{AB} coincides with the dislocated wave incident from the right plus outgoing waves only. Exactly in the same way as in I, this leads to

$$\frac{c_n}{J_{m_+}(\beta)} = (-i)^{m_+}. \quad (3.23)$$

It is important, in order to use this result, that either the coefficient q_+^2 for positive δ or q_-^2 for negative δ in Eq. (3.7) be equal to k^2, and that they be associated in each case to m_+. Otherwise, it would have been impossible to recover the dislocated wave, which is a crucial physical requirement for the solution because we need $q=k$ to be a possible result of the factorization. This fact fully justifies the factorization in Eq. (3.7). We do not display the systems of equations, neither their solutions, which are not too illuminating. We use the capability of MATHEMATICA [7] for symbolic and numerical calculations to get the coefficients. We insist on the fact that the solution may be inaccurate at a few wavelengths away from the vortex because of the approximate character of factorization (3.7).

Let us discuss the asymptotic behavior of the solution for $r \to \infty$. The case of η_{AB} is completely similar to the shallow water case. Indeed, the index of the Bessel function, $m_+(n) = \sqrt{n^2 + (\mathrm{Const.}) \times n}$, has exactly the same structure as $m(n)$ in I. An important consequence is that the parameter $\alpha = \beta M(c_\phi/c_g)$ has the same physical significance as in the shallow water (or acoustics) case: it quantifies the dislocation of the wave fronts in the forward direction at large distances from the vortex. Other results may also be transposed in a straightforward fashion, and the asymptotics of $\eta_{AB}(r,\theta)$ for large r is still given by Eq. (4.19) of I, with the proviso that the function $G(\theta, -\pi/2)$ takes into account new definition (3.16) of α.

The asymptotics of η_R depends on the sign of δ. If δ is positive, the hyperbolic Bessel function does not contribute to the scattered far field because [8] $K_p(z) \sim e^{-z}/\sqrt{z}$ for large z. We thus get the same behavior as in I. In the next section, we will compare the correction to the Aharonov-Bohm scattering amplitude in the case of shallow water waves, given by Eq. (4.22) of I, and the correction for dispersive water waves, which reads

$$G_{DW}(\theta, -\pi/2) + 2\sum_n \frac{d_n}{H^1_{m_+}(\beta)} e^{in\theta}(-i)^{m_+}, \quad (3.24)$$

where G_{DW} is just the function G of paper I with α redefined by Eq. (3.16).

If δ is negative, we must take into account the two outgoing Hankel functions, so that the correction for dispersive water waves now reads

$$G_{DW}(\theta,-\pi/2)+2\sum_n\left[\frac{d_n(-i)^{m_+}}{H^1_{m_+}(\beta)}+\frac{e_n(-i)^{m_-}}{(|\delta|-1)^{1/4}H^1_{m_-}(\beta\sqrt{|\delta|-1})}\right]e^{in\theta}. \quad (3.25)$$

IV. NUMERICAL EXAMPLES

The solution to the scattering problem of surface waves by a uniform vertical vortex depends on four dimensionless parameters. A first set includes the dimensionless vortex radius $\beta\gg 1$ and the dislocation parameter $\alpha=\beta M(c_\phi/c_g)$, which quantifies the wave front dislocation. They already appeared in I, with the same definitions and physical interpretations. A third parameter is the dimensionless capillary length $\ell\equiv kl_c$, and the last one is the dimensionless depth kh. In order to simplify somewhat the discussion, we use the single dimensionless parameter δ, defined in Eq. (2.27), in place of the depth and capillary length. As an example, we take $\delta=8$ that may correspond, for example, to $h=l_c$, $kh=\sqrt{3}/4$, and $\delta=-8$, which may correspond to $h=3l_c$ and $kh=3/4$. In both cases, the hyperbolic tangent in the water waves dispersion relation [5] is approximated to better than five percent by the two leading terms, the ones we are keeping in its series expansion.

Scaling radial distance with the vortex radius, $r'\equiv r/a$, the analytical expression of the surface displacement is summarized as follows, depending on the sign of δ. Inside the vortex $(0<r'\leqslant 1)$ we have $\eta_1=\mathrm{Re}\,\eta_c$.

For *positive* values of δ,

$$\eta_c=\sum_n\left(a_n\frac{J_n(\tilde{\phi}_n r')}{J_n(\tilde{\phi}_n)}+b_n\frac{I_n(\varphi_n r')}{I_n(\varphi_n)}\right)e^{i(n\theta-\nu t)}, \quad (4.1)$$

where we have defined the following dimensionless wave numbers:

$$k_n a\equiv\tilde{\phi}_n=\beta\sqrt{\frac{\delta}{2}}\left[-1+\sqrt{1+4\frac{1+\delta}{\delta^2}\left(1-n\frac{\alpha}{\beta^2}\frac{2+\delta}{1+\delta}\right)^2}\right]^{1/2}, \quad (4.2)$$

$$\kappa_n a\equiv\varphi_n=\beta\sqrt{\frac{\delta}{2}}\left[1+\sqrt{1+4\frac{1+\delta}{\delta^2}\left(1-n\frac{\alpha}{\beta^2}\frac{2+\delta}{1+\delta}\right)^2}\right]^{1/2}. \quad (4.3)$$

For *negative* values of δ,

$$\eta_c=\sum_n\left(a_n\frac{J_n(\phi_n r')}{J_n(\phi_n)}+b_n\frac{J_n(\tilde{\varphi}_n r')}{J_n(\tilde{\varphi}_n)}\right)e^{i(n\theta-\nu t)}, \quad (4.4)$$

with new dimensionless wave numbers:

$$k_n a\equiv\phi_n=\sqrt{\frac{|\delta|}{2}}\beta\left[1-\sqrt{1-4\frac{|\delta|-1}{\delta^2}\left(1-n\frac{\alpha}{\beta^2}\frac{|\delta|-2}{|\delta|-1}\right)^2}\right]^{1/2}, \quad (4.5)$$

$$\kappa_n a\equiv\tilde{\varphi}_n=\sqrt{\frac{|\delta|}{2}}\beta\left[1+\sqrt{1+4\frac{|\delta|-1}{\delta^2}\left(1-n\frac{\alpha}{\beta^2}\frac{|\delta|-2}{|\delta|-1}\right)^2}\right]^{1/2}. \quad (4.6)$$

Outside the vortex $(r'>1)$ $\eta_1=\mathrm{Re}(\eta_{AB}+\eta_R)$, where, whatever the sign of δ,

$$\eta_{AB}=\sum_n(-i)^{m_+}J_{m_+}(\beta r')e^{i(n\theta-\nu t)}. \quad (4.7)$$

For *positive* values of δ,

$$\eta_R=\sum_n\left(d_n\frac{H^1_{m_+}(\beta r')}{H^1_{m_+}(\beta)}+e_n\frac{K_{m_-}(\beta\sqrt{1+\delta}r')}{K_{m_-}(\beta\sqrt{1+\delta})}\right)e^{i(n\theta-\nu t)}, \quad (4.8)$$

where we have used the fact that

$$qa=\beta\sqrt{1+\delta}. \quad (4.9)$$

For *negative* values of δ,

$$\eta_R=\sum_n\left(d_n\frac{H^1_{m_+}(\beta r')}{H^1_{m_+}(\beta)}+e_n\frac{H^1_{m_-}(\beta\sqrt{|\delta|-1}r')}{H^1_{m_-}(\beta\sqrt{|\delta|-1})}\right)e^{i(n\theta-\nu t)}, \quad (4.10)$$

where we have used the fact that

$$qa=\beta\sqrt{|\delta|-1}. \quad (4.11)$$

We do not give the details of the calculations of the coefficients a_n, b_n, d_n, and e_n. Although their definitions are far more complicated, their behavior is very similar to the previous case (see I). As an illustration, absolute values of those coefficients are plotted in Fig. 1 as a function of their index, and their rapid decrease suffices to indicate (simple) convergence of the corresponding series.

Since convergence of the series expansions for η_{AB} and η_R is not uniform, the number of terms to keep in the numerical evaluation of the infinite series depends on the value of r'. In practice, the convergence of the series is comparable to the case of I, and we use roughly the same number of terms. We compute the patterns of the surface displacement in the region $|x'|,|y'|\leqslant 5[(x',y')=(r'\cos\theta,r'\sin\theta)]$ by the finite sum of Eqs. (4.1), (4.4), (4.8), and (4.10) with $|n|\leqslant 50$ for $\beta=10$ and $|n|\leqslant 30$ for $\beta=5$, but we keep more terms, $|n|\leqslant 90$ in Eq. (4.7). All calculations were performed using MATHEMATICA [7].

Let us first consider the case of positive δ. Figs. 2(a), and 2(c) show the resulting displacements for $\delta=+8$, $\beta=5$, and $\alpha=1,2$, and Figs. 3(a), and 3(c), for the same values of α and δ, but $\beta=10$. The dislocation of the incident wave fronts by

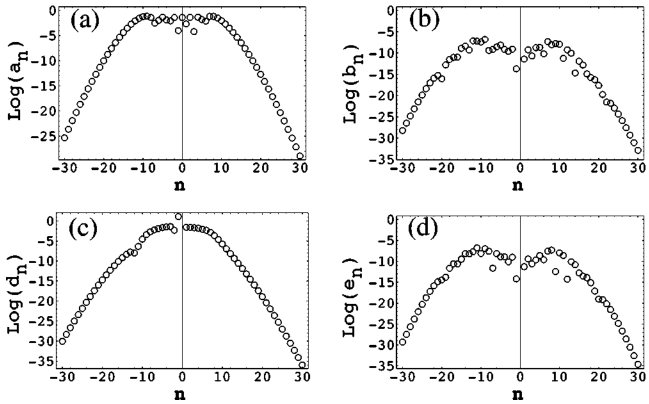

FIG. 1. Absolute magnitude of the neperian logarithm of coefficients a_n (a), b_n (b), d_n (c), and e_n (d), versus n in a log-linear plot. The parameters used in the calculations are $\alpha=1$ and $\beta=10$. On the same graph, we display both cases of positive $\delta=+8$ (dots) and negative $\delta=-8$ (empty circles). Note the asymmetry with respect to $n\rightarrow -n$, and the very quick decrease of the coefficients.

an amount equal to α is clearly visible. The outward traveling scattered wave is also visible. The interference patterns between scattered and incident wave is very similar to the corresponding pictures of I, for the same values of β and α. This is confirmed by the comparison of the scattering cross section displayed in Figs. 4(a), and 4(c), as discussed below. Taking into account the dispersion greatly modifies the numerical value of α, but other corrections are small in the case of positive δ.

In the case of negative δ, the interference pattern is strongly modified. This is illustrated in Figs. 2(b) and 2(d),

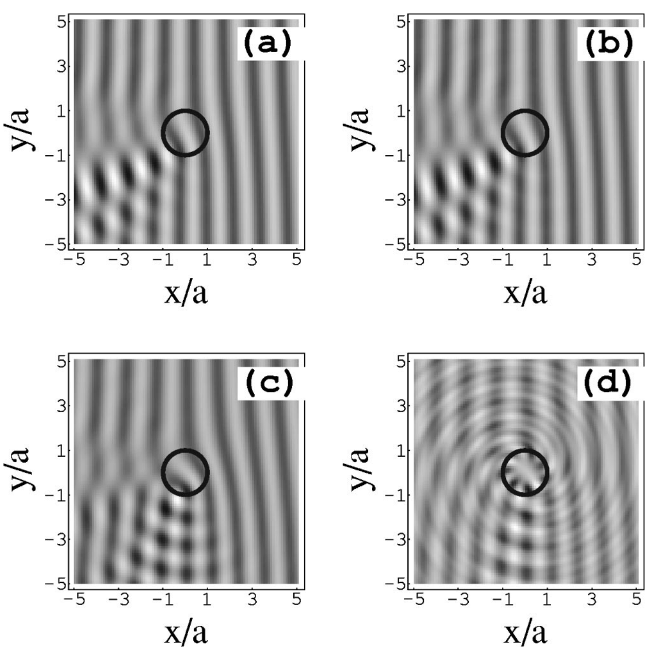

FIG. 2. Density plot of the surface elevation for the total wave patterns for $\beta=5$, and several values of α and δ. Respectively, $\alpha=1$, $\delta=+8$: (a), $\alpha=1$, $\delta=-8$: (b), $\alpha=2$, $\delta=+8$: (c), and $\alpha=2$, $\delta=-8$: (d). The gray scale is linear with surface amplitude (arbitrary units). The dark ring indicates the vortex location, and the vortex rotates counterclockwise. The wave is incident from the right. The box size is 10×10 in units of a, the vortex radius.

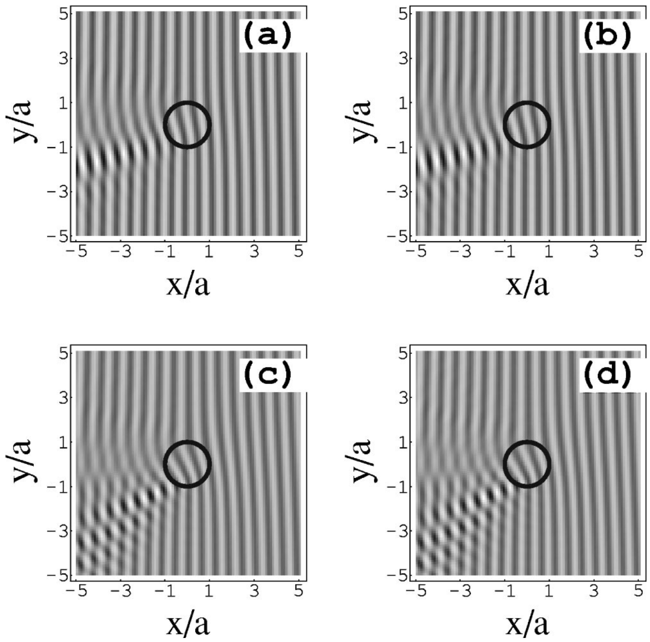

FIG. 3. Same as Fig. 2, for $\beta=10$.

where we plot the surface displacement for $\delta=-8$, $\beta=5$, and $\alpha=1,2$. The spiral wave, which is clearly seen for $\alpha=2$ [Fig. 2(d)], is very different from the corresponding figure of I [see Fig. 2(d) of I]. For larger values of β shown in Figs. 3(b) and 3(d) for which $\beta=10$ with the same values for δ and α as before, the pictures are much more similar to the shallow water case.

Those results obtained at a finite distance from the vortex are fully confirmed asyptotically far from it by the plots of the absolute value of the correction to the Aharonov-Bohm scattering amplitude in both cases. The dashed line in Fig. 4 shows this correction in the shallow water (nondispersive) case, and the solid line shows the same correction in the dispersive case, for a positive $\delta=+8$ in Figs. 4(a) and 4(c) and a negative $\delta=-8$ in Figs. 4(b) and 4(d). In the positive δ case, both corrections are almost the same, in agreement with the results at finite distance shown in Figs. 2 and 3. In the negative δ case, apart for the smallest values of α [see Fig. 4(b)], the corrections are markedly different. In the negative δ dispersive case, the scattering is much more isotropic, and very different in amplitude. An obvious, but somewhat formal explanation of this difference is the supplementary function $H^1_{m_-}(\beta\sqrt{|\delta|-1})$ in Eq. (3.25) compared to Eq. (3.24). Also, the index of this function is $m_-(n)$ that takes imaginary values for small *positive* n, rather than for small *negative* n as $m_+(n)$. This implies that the partial amplitudes for $\exp(-in\theta)$ and $\exp(in\theta)$ are much more similar to each other than in the shallow water case, for which m_- is absent, and also more similar than in the case of positive δ, where the decrease of the corresponding function is exponential. The appearance of an algebraically ($\propto 1/\sqrt{r}$) decreasing amplitude associated to m_- in the negative δ case restores the symmetry of the scattered wave.

A more physical explanation is as follows: Consider a plane wave incident from the right on a counterclockwise vortex, as in Figs. 2 and 3. Above the vortex, the wave front velocity is increased by advection, whereas it is decreased

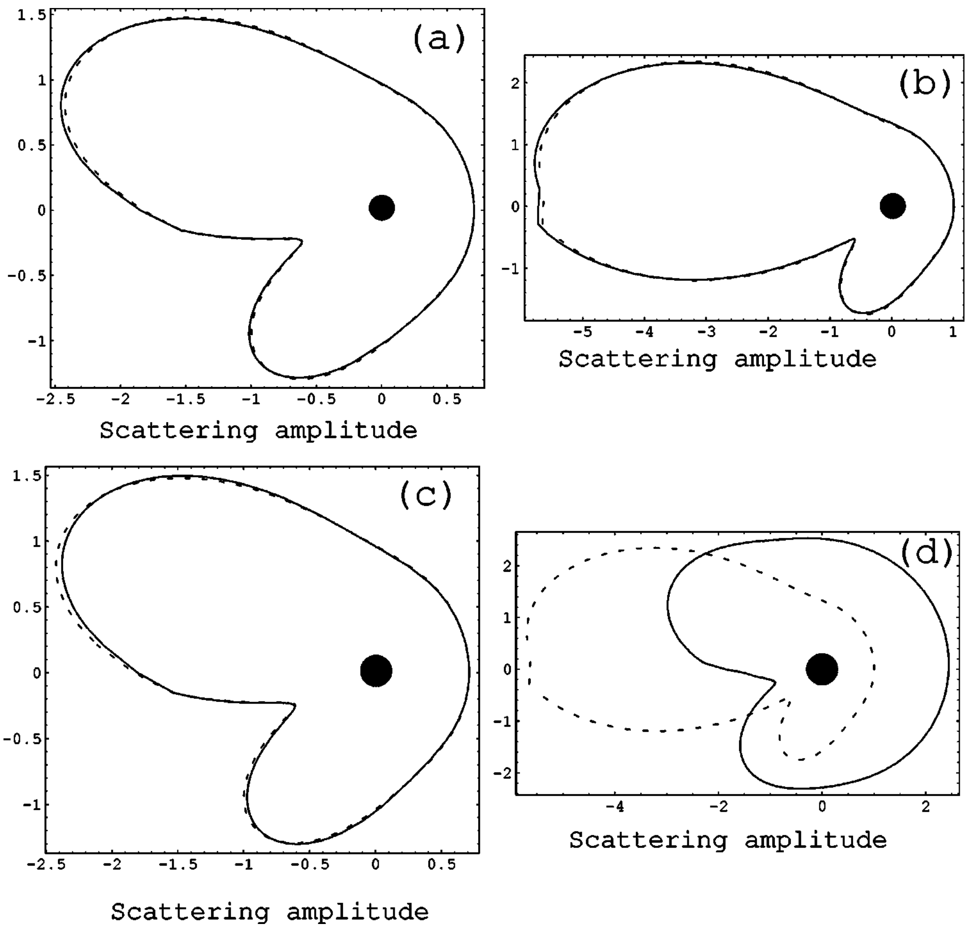

FIG. 4. Polar plot of the absolute value of the correction to the Aharonov-Bohm (i.e., point) scattering amplitude, in the case of nondispersive waves (dashed line) and in the case of dispersive waves (solid line), for $\beta=5$ and, respectively: $\alpha=0.25$, $\delta=+8$: (a), $\alpha=0.25$, $\delta=-8$: (b), $\alpha=0.5$, $\delta=+8$: (c), and $\alpha=0.5$, $\delta=-8$: (d). In this last case the dispersive wave scatters very differently from the nondispersive wave. The vortex location is marked by the large dot; the vortex rotates counterclockwise.

below the vortex. Consequently, the wave fronts should bend towards the bottom of the picture, as they do. The other effect of the vortical flow is to add a wavelength below the vortex, which means that the wave number k decreases below the vortex. Since kh is assumed to be small, for positive δ the phase velocity increases with k. The phase velocity is thus smaller below the vortex than above, which enhances the bending of the wave fronts, and reinforces the strong asymmetry in the interference pattern of Figs. 2(a) and 2(c) and Figs. 3(a) and 3(c), and in the scattering amplitude of Figs. 4(a) and 4(c). For negative δ, the phase velocity *decreases* with k, and the effect of the dislocation is to make the phase velocity smaller for the part of the wave front above the vortex. This effect balances the effect of advection, and we understand why the interference pattern [see Figs. 2(b) and 2(d) and Figs. 3(b) and 3(d)] and the scattering amplitude [see Fig. 4(d)] are much more symmetric than in the positive δ case, or than in the nondispersive ($\delta=0$) case. It is also reasonable that this effect should be more important for $\beta=5$ than for $\beta=10$, because the relative variation in k due to the dislocation is greater in the former case. The spiral waves are observed for negative δ because the interference pattern almost keeps rotational symmetry while smoothing the wave front dislocation in the forward direction.

As a last illustration, we compare the wave patterns predicted by the shallow water approximation and by its first correction in powers of the fluid depth in an experimentally accessible situation. We suppose the fluid to be pure water, of depth 1 mm; the vortex radius is 1 cm and the wavelength is 2 cm. Thus $kh=\pi/10$, and the approximation of the dispersion relation is excellent. The price to pay is the rather small value $\beta=\pi$. We take the vortex circulation such that $\alpha=1$ in the shallow water approximation, for which $c\equiv\sqrt{gh}=9.9$ cm/s. Taking the dispersion into account, we get $\delta=1.4$, $c_\phi=13.0$ cm/s, $c_{\mathrm{g}}=18.4$ cm/s, and $\alpha=0.41$. The difference in the respective numerical values of α is the leading effect. The result is shown in Fig. 5. To obtain quantitative agreement with an experimental situation, it may be sufficient to keep the shallow water approximation, but with the actual value of α obtained in the dispersive case [Eq. (3.16). It should be interesting to use a fluid with a very small

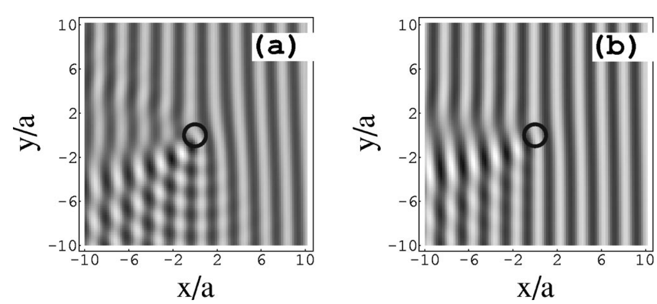

FIG. 5. Density plot of the surface elevation for the total wave pattern calculated in the shallow water approximation (a) and to first order in fluids depth (b). The gray scale is linear with surface amplitude (arbitrary units). The dark ring indicates the vortex location, and the vortex rotates counterclockwise. The incident wave comes from the right. The box size is 20×20 in units of a, the vortex radius. $\beta=\pi$ in both cases, but $\alpha=1$ in the shallow water case (a), and $\alpha=0.41$, $\delta=1.4$ in the dispersive case (b).

surface tension in order to obtain a negative value of δ while keeping a small value of kh for which the wave pattern should be extremely different from the shallow fluid layer approximation.

V. CONCLUDING REMARKS

We have computed the surface displacement due to a dispersive surface wave interacting with a vertical vortex when the vortex core performs solid body rotation; the wavelength is small compared to the vortex core radius and the particle velocities associated with the wave are small compared to the particle velocities associated with the vortex. When the parameter $\alpha = \nu\Gamma/2\pi c_{\phi} c_{\mathrm{g}}$ is of the order of 1 or bigger, the wave fronts become dislocated. This parameter depends, in the dispersive case, both on the phase and group velocity of the wave, and tends smoothly toward the result of I in the nondispersive limit. We thus give a proof in a perturbative fashion of the heuristic derivation of Berry *et al.* [3].

Formally, we proceed perturbatively around the shallow water limit to obtain a fourth-order partial differential equation for the surface elevation associated with the surface wave. However, apart from some technical details, the solution is very similar to the nondispersive case. The scattered waves interact strongly with the dislocated wave fronts and produce interference patterns. A dimensionless parameter δ quantifies the relation between fluid layer depth h and capillary length l_c. When it is positive ($h < \sqrt{3} l_c$) the wave pattern is similar to the shallow water case. When it is negative, for large values of the circulation, the wave pattern is very different.

We hope that the calculations in the dispersive case will help the comparison with experiments. Our calculations are valid when the approximation $\tanh kh \simeq kh - (kh)^3/3$ holds. This is a restrictive condition, but we believe that once dispersion is correctly taken into account in the definition of α, the wave pattern given by the nondispersive case should be roughly similar to the observations. Some discrepancies are expected when the fluid depth is greater than the capillary length ($h > \sqrt{3} l_c$), but in this case it is necessary to correctly approximate the hyperbolic tangent by the first two terms of the series, and in practice the effect should be observable for a fluid of small surface tension (thus small capillary length) only. However, it is quite possible that effects such as the spiral patterns will persist qualitatively in the case of deep water, when our perturbative approach will no longer hold.

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